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Optimum Translation and the Brachistochrone

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1. Introduction

Requirements for a manned lunar landing include the ability to descend to a low altitude, hover for a time sufficient to examine the potential landing site, and translate if necessary to a new landing site. In Ref. 1, several possible translation maneuvers are analyzed from the point of view of fuel consumption. In this paper we determine the maneuver which results in the least possible fuel consumption. The solution is effected by transforming the problem into a form equivalent to the brachistochrone with limited slope and applying standard techniques of the calculus of variations.

It will be assumed that the maneuver is horizontal and perpendicular to the gravitational acceleration g, which is constant in magnitude and direction. It will further be assumed that the thrust acceleration a_T is bounded in magnitude. In particular, let μ denote the horizontal component of the thrust to weight ratio a_T/g and M the maximum value of μ. Since for horizontal motion the vertical component of the thrust acceleration is g, the total acceleration due to thrust is limited as follows:

$$0 \leqslant a_T \leqslant g(M^2 + 1)^{1/2} \tag{1}$$

The assumption of the limit on thrust acceleration, rather than thrust, results in analytical simplifications which enable closed form solutions to be obtained easily.

2. The Variational Problem

The equations of motion are

$$\ddot{x} = \mu g \qquad \ddot{y} = 0 \tag{2}$$

where g is the vertical component of the thrust acceleration and μg is the horizontal component. The thrust acceleration magnitude is then

$$a_T = g(1 + \mu^2)^{1/2} \tag{3}$$

and the characteristic velocity is

$$\Delta V = \int_0^{t_f} a_T\, dt = g \int_0^{t_f} (1 + \mu^2)^{1/2}\, dt \tag{4}$$

where t_f is the time of flight.

The problem is to determine that function $\mu(t)$ which minimizes (4) subject to (1), (2), and the end conditions:

$$\begin{aligned} t = 0: &\qquad x = 0 \qquad \dot{x} = 0 \\ t = t_f: &\qquad x = D \qquad \dot{x} = 0 \end{aligned} \tag{5}$$

That is, the vehicle translates through the distance D, beginning and ending in a stationary hovering position.

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The problem can be transformed into the classical first problem of the calculus of variations by treating distance as the independent and velocity as the dependent variable. We define $\xi = 2x/D$, $u = v^2/gD$, $v = \dot{x}$ and Eq. (4) becomes

$$\frac{\Delta V}{(gD)^{1/2}} = \frac{1}{2}\int_0^2 \left[\frac{1 + u'^2}{u}\right]^{1/2} d\xi \tag{6}$$

where $u' = du/d\xi = \mu$. The inequality constraint (1) can be expressed by the equality,[2]

$$\epsilon^2 = M^2 - u'^2 \tag{7}$$

where ϵ is a new state variable. Incorporating the constraint (7) by the method of Lagrange multipliers leads to the following integral to be minimized:

$$\frac{\Delta V}{(gD)^{1/2}} = \int_0^1 \left[\left(\frac{1 + u'^2}{u}\right)^{1/2} + \lambda(\epsilon^2 - M^2 + u'^2)\right] d\xi = \int_0^1 F(x,u,u')d\xi \tag{8}$$

where the integration interval has been reduced to half the distance to take advantage of symmetry. The boundary conditions are

$$\begin{aligned} \xi = 0: &\qquad u = 0 \\ \xi = 1: &\qquad u \text{ not prescribed} \end{aligned} \tag{9}$$

That is, the problem of translating a distance D, beginning and ending at zero velocity, is equivalent to that of translating a distance $D/2$ beginning at rest and with arbitrary final velocity.

Equations (8) and (9) are identical to those obtained from the brachistochrone problem, where it is desired to find the shape $y(x)$ of a frictionless wire joining two points, such that a particle sliding along the wire under the influence of gravity alone will travel between the points in minimum time. To complete the analogy $\xi \equiv 2x/D$, $u \equiv 2y/D$ and $\Delta V \equiv gT$, where T is the duration of the motion. The constraint corresponds to a limit on the maximum slope the wire is allowed to assume.

3. Solution of the Variational Problem

The solution is obtained by direct application of the methods of the calculus of variations described in Ref. 2. In particular it may be shown that either $\epsilon = 0$ and the acceleration is maximum or minimum, or $\lambda = 0$ and an intermediate level prevails. The corner conditions show that λ and u' are continuous at the transition and the free boundary conditions yield $u' = 0$ at $\xi = 1$, where the velocity is maximum. The transition from maximum to intermediate thrust acceleration is determined from the Weierstrass condition.

The resulting optimum thrust acceleration is given as a function of the instantaneous velocity by

$$\begin{aligned} \mu = u' &= M & (u < u_s) \\ &= \left(\frac{u_m}{u} - 1\right)^{1/2} & (u > u_s) \end{aligned} \tag{10}$$

where the maximum value of the velocity variable u is

$$u_m = \left[\frac{1}{M} + \cot^{-1}\frac{1}{M}\right]^{-1} \tag{11}$$

and the switching condition is

$$u_s = u_m/(1 + M^2) \tag{12}$$

The trajectory is

$$\begin{aligned} \xi &= u/M \qquad (u < u_s) \\ &= 1 - u_m\left\{\left[\frac{u}{u_m}\left(1 - \frac{u}{u_m}\right)\right]^{1/2} + \cos^{-1}\left[\frac{u}{u_m}\right]^{1/2}\right\} \qquad (u > u_s) \end{aligned} \tag{13}$$

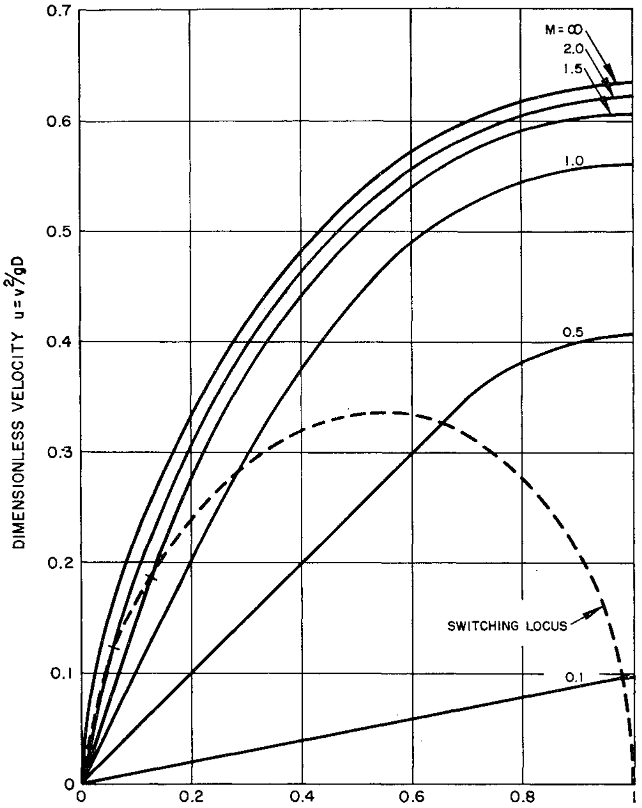

Fig. 1 Velocity-range trajectory.

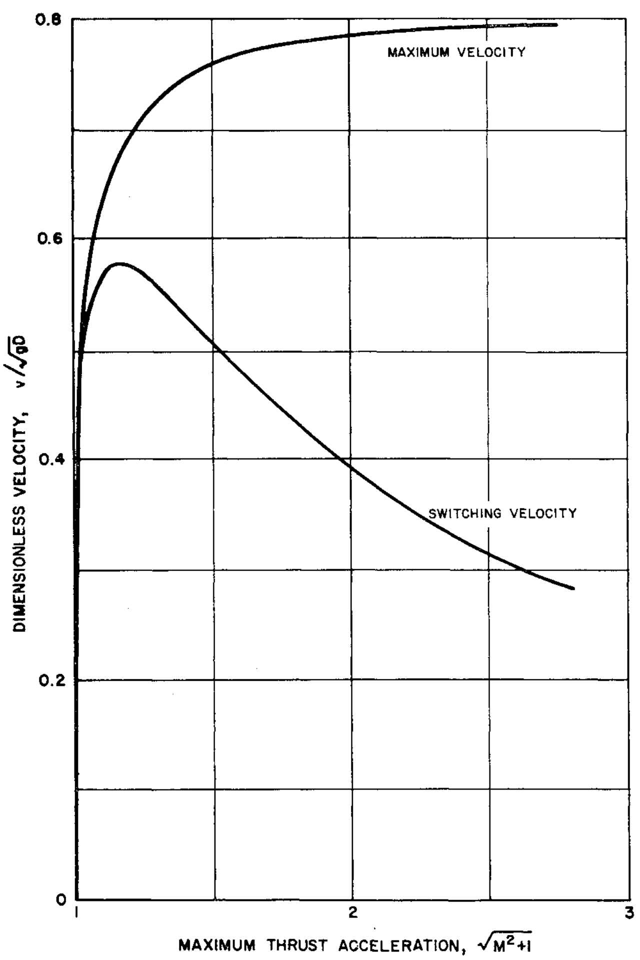

Fig. 2 Maximum velocity vs acceleration limit.

In Fig. 1 this trajectory is shown for various M, and Fig. 2 gives the peak and switching velocities from (11) and (12).

Integration of (10) yields the time history and the duration t_f of the maneuver

$$\frac{1}{2}\left(\frac{g}{D}\right)^{1/2} t_f = \left(\frac{1 + M^2}{M^2} u_m\right)^{1/2} \tag{14}$$

The characteristic velocity from (6) is then

$$\Delta V = 2(gD)^{1/2}\left(\frac{1}{M} + \tan^{-1}M\right)^{1/2} \tag{15}$$

which is plotted in Fig. 3.

4. Suboptimal Control Policies

For comparison, results are also shown for two suboptimal control laws discussed in Ref. 1. In the first, the acceleration is maintained at a constant value throughout the maneuver. In this case it is easily shown that if $M \geq 1$, the optimum value of the horizontal acceleration is one $g(\mu = 1)$, and $\Delta V = 2(2)^{1/2}(Dg)^{1/2}$. It is interesting to note that this

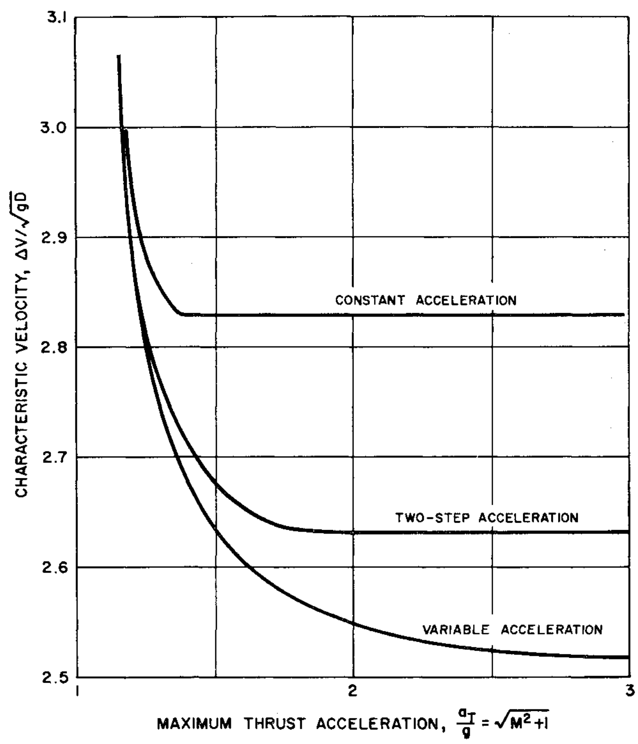

Fig. 3 Minimum characteristic velocity vs acceleration limit.

is the same as the result for the optimum constant velocity maneuver, initiated and terminated by impulses. For smaller values of M, maximum acceleration is maintained, with the result

$$\Delta V = 2\left[gD\,\frac{M^2 + 1}{M}\right]^{1/2} \tag{16}$$

In the second suboptimal policy, the vehicle is accelerated with constant horizontal component μg for time t_1. A coast period ($\mu = 0$) for duration t_2 follows, after which slow-down occurs in t_1 sec. For this maneuver, the optimum switching times t_1 and t_2 are given by

$$\begin{aligned} t_1 &= \left(\frac{D}{g}\right)^{1/2}\{\mu[2(1 + \mu^2)^{1/2} - 1]\}^{1/2} \\ t_2 &= 2t_1[(1 + \mu^2)^{1/2} - 1] \end{aligned} \tag{17}$$

If $M \geqslant 3^{1/2}$, then $\mu_{\text{opt}} = 3^{1/2}$ and $\Delta V = 2\,(3)^{1/4}(gD)^{1/2}$. For smaller values of M, the acceleration during the first phase is maximum and the result is

$$\Delta V = 2(gD)^{1/2}\left[\frac{2(1+M^2)^{1/2}-1}{M}\right]^{1/2} \tag{18}$$

Equations (17) and (18) are plotted in Fig. 3.

The horizontal translation itself forms a subclass with respect to all possible maneuvers between the required end conditions. The best such maneuver (two impulses) results in $\Delta V/(gD)^{1/2} = 2.0$. The fuel penalty for restricting the maneuver to be horizontal is thus about 25%.

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Technique for Minimizing the Specific Weight of a Finned Beryllium Space Radiator

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Nomenclature

- B = total width of tube with fins, ft
- C = specific heat, Btu/lb, °R
- D = tube diameter, ft
- ϵ = emissivity
- k = thermal conductivity, Btu/hr, ft, °R
- K = constant
- l = length along tube, ft.
- L = tube length, ft
- Q_r = $f(V)$ = rejected power, Btu/hr
- t = fin thickness, ft
- $\bar{T}$ = temperature, °R
- T_w = temperature of tube wall, °R
- T = average radiating temperature, °R
- T_{w1} = temperature of tube wall at tube inlet, °R
- T_{w_2} = temperature of tube wall at tube outlet, °R
- V = velocity of liquid metal in tube, ft/hr
- W = $F(V)$ = total weight, lb_f
- X = length from center of tube to tip of fin, ft
- σ = Stefan-Boltzmann constant, Btu/hr, ft², °R⁴
- ρ = density of liquid metal, lb/ft³

SINCE in studying large spacecraft power plants, it has been found that the radiator constitutes a major portion of the total weight of the powerplant,[1] the importance of minimizing the specific weight of the radiator is apparent. Beryllium is a very attractive material for high-temperature, finned space radiators.[2] Reference 2 also indicates that tube diameter, fin thickness, and length should be small so that the specific weight (lb/kw of heat rejected) of the radiator may be low. Reference 2 deals with the rectangular fin and assumes that the thermal conductivity of the fin material is independent of the temperature. Reference 3 indicates that a triangular fin is more effective in rejecting heat than a rectangular one, and hence such a fin is better suited for a space radiator.

This article, therefore, discusses a beryllium radiator having a fin that is triangular in cross section and containing a wettable liquid metal in the tube for power plant cooling, and it presents a simple technique for minimizing the specific weight of a small-diameter beryllium tube with a short and thin triangular fin. This technique permits thermal conductivity to vary with temperature and can be used for any fin profile. It also allows the temperature of the base of the fin to vary, accounts for the weight of the radiator fluid and pump, and includes the effect of these parameters in the optimization of the specific weight of the radiator.

The principal problem is to express the heat rejected by the radiator as a function of the fluid velocity, since the combined weight of the radiator and pump can be expressed as a function of the fluid velocity by fundamental and well-known equations. By using the generally accepted value of 0.8 for emissivity of the radiator surface, it is shown that the relationship between the temperature of the tube wall and the average radiating temperature of the finned tube at temperatures above 800°R is almost linear. If, as a good approximation, this relationship is assumed to be linear, the equation for the specific weight of the radiator becomes quite simple and its minimum point can be found easily.

Assumptions

In treating this subject, the following assumptions have been made: 1) the radiating surface is a gray surface; 2) the diameter of the tube is such that the temperature of the tube around the circumference is constant, and the radiant interaction between fin and tube is negligible; 3) heat flow by conduction is one-dimensional; 4) resistance to the flow of heat from the liquid metal to the outer surface of the tube is negligible; 5) the radiating area is a flat surface, and any energy incident on it is not considered.

Analysis

The energy lost by the liquid metal as it passes through the finned tube is

$$Q_r = \frac{\pi D^2 V\rho C(T_{W_1} - T_{W_2})}{4} \tag{1}$$

An energy balance for the nodal point 1 in Fig. 1 gives

$$Q = \frac{k_{0,1}t_{0,1}l}{\Delta X}(T_W - T_1) - \frac{k_{1,2}t_{1,2}l}{\Delta X}(T_1 - T_2) - 2\sigma\epsilon\Delta X\, lT_1^4 \tag{2}$$

Equations are set up for the other nodal points. In the steady state, Q is equal to zero, and the unknown temperatures T_1, T_2, T_3, and T_4 are solved by relaxation methods. The values of T_W and ϵ are fixed, but that of k varies with temperature. By using the typical temperature curve in Fig. 1, one finds that the energy balance is

$$\sigma\epsilon l\Delta X(T_1^4 + T_2^4 + \ldots T_4^4) + \frac{\sigma\epsilon lDT_W^4}{2} = \sigma\epsilon Xl\bar{T}^4 \tag{3}$$

The average radiating temperature $\bar{T}$ at any position of the finned tube is obtained from this equation. By varying the value of T_W, Eqs. (2) and (3) give the relationship between T_W and $\bar{T}$ for a particular fin and tube (see Fig. 2). The tube diameter is $\frac{1}{16}$ in., the emissivity is 0.8, and the fins have a triangular cross section with a root thickness of 0.004 in. The length of the fin varies from 0.5 to 2.0 in. Similar curves were obtained for tubes having diameters of $\frac{1}{8}$ in. and $\frac{1}{4}$ in., with a fin root thickness of 0.008 in. and the lengths indicated above. From Fig. 2, $dT_w/d\bar{T}$ can be considered constant for a particular curve. The energy balance for an infinitesimal element of the finned tube is

$$-\pi D^2 V\rho C(dT_W)/4 = 2\sigma\epsilon B(dl)\bar{T}^4 \tag{4}$$

Since $dT_W/d\bar{T} = K$ = constant, integrating and solving for $\bar{T}$ in Eq. (4) gives

$$\bar{T} = \left(\frac{\bar{T}_1^3\pi D^2 V\rho CK}{\pi D^2 V\rho CK + 24\sigma\epsilon BL\bar{T}_1^3}\right)^{1/3} \tag{5}$$

Integrating $dT_W/d\bar{T} = K$ gives

$$T_{W_1} - T_{W_2} = K(\bar{T}_1 - \bar{T}) \tag{6}$$

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